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CONSTITUTIVE RELATIONS
FOR FINITE ELASTIC–INELASTIC STRAINS

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The evolutionary constitutive elastic-inelastic relation with its compatible objective derivative is derived in general form using the kinematics of superposition of small elastic and inelastic strains on finite elastic–inelastic strains. The equation is rendered concrete using the elastic law for a slightly compressible material.

Key words: *elastic–inelastic behavior, finite strains, slight compressibility, evolutionary constitutive equations.*

1. Preliminary Information. Using three configurations: an initial configuration $æ_0$, a current configuration $æ$, and an intermediate configuration $æ_*$ close to the current configuration and employing the kinematics of superposition of small strains (position gradients) on finite strains, Novokshanov and Rogovoi [1] derived constitutive equations for finite elastic strains of a simple material relative to the intermediate configuration.

According to the Celerier–Richter theorem or the Noll reduction theorem, the constitutive equation for a simple material that satisfies the objectivity principle is written as (see [2])

$$T = R \cdot \tilde{g}_1(U) \cdot R^{\mathrm{t}}, \tag{1.1}$$

where T is the true stress tensor; R and U are the orthogonal tensor and the symmetric positive definite pure-strain tensor in the polar decomposition of the position gradient $F = R \cdot U$; $\tilde{g}_1(U)$ is the material response to pure strain. Relation (1.1) can be written in several equivalent forms [1], in particular,

$$T = J^{-1} F \cdot \tilde{g}_6 \cdot F^{\mathrm{t}}, \tag{1.2}$$

where $J = I_3(F)$ is the third basic invariant F, which defines the relative volume change; and $\tilde{g}_6$ is the material response function. In [1], the function $\tilde{g}_6$ is linked to $\tilde{g}_1$ by the relation $\tilde{g}_1 = J^{-1} U \cdot \tilde{g}_6 \cdot U$. For the intermediate configuration $æ_*$ close to the current configuration, the constitutive equation (1.2) is written as

$$T = [1 - I_1(e)]T_* + h \cdot T_* + T_* \cdot h^{\mathrm{t}} + \tilde{L}_6^{IV} \cdot\cdot\, e. \tag{1.3}$$

Here T_* is the stress reached in the configuration $æ_*$ (the initial stress for this configuration), $h = (\overset{æ_*}{\nabla} \boldsymbol{u})^{\mathrm{t}}$ is the gradient (relative to the configuration $æ_*$ $\boldsymbol{u}$) of the vector of small displacements that transform the intermediate configuration to the current configuration, $e = (h + h^{\mathrm{t}})/2$ is the small-strain tensor relative to the configuration $æ_*$, $d = (h - h^{\mathrm{t}})/2$ is the small-rotation tensor, $\tilde{L}_6^{IV}$ is the fourth-rank tensor (generally, anisotropic) which defines the elastic response of the material to small strains relative to the intermediate configuration.

The approximate relation (1.3) can be made exact by dividing by the increment in the time of transition from the intermediate to the current configuration and passing to the limit, i.e., by letting the intermediate configuration to the current configuration. As a result, we have the evolutionary equation

$$T^{\mathrm{Tr}} = \tilde{L}_6^{IV} \cdot\cdot\, \dot{e} \tag{1.4}$$

with the Truesdell objective derivative.

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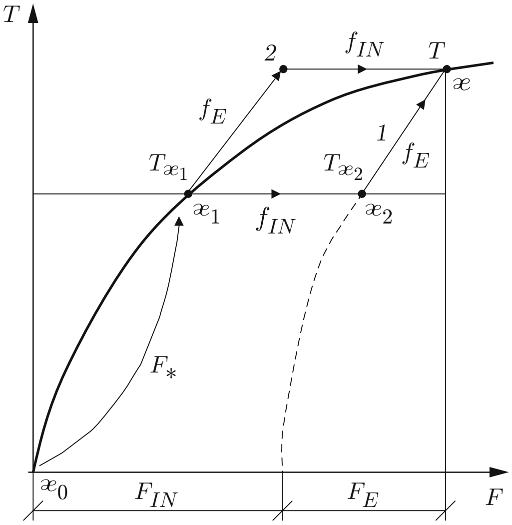

Fig. 1

The configuration $æ_*$ is obtained from the current configuration (unknown before the solution of the problem) by small elastic unloading, and if the process is purely elastic, it coincides with the configuration $æ_1$ attained at the end of the previous loading step. If the process is elastic–inelastic, the configuration $æ_*$, as shown in [1], is also uniquely derived from the known configuration $æ_1$. Using relations (1.3) and (1.4) and considering an elastoplastic process to be elastic with the reference configuration $æ_2$ obtained from the configuration $æ_1$ by a small plastic rotation d_P, Novokshanov and Rogovoi [1] derived evolutionary constitutive equations for large elastoplastic strains (finite elastic and finite plastic strains) for an arbitrary elastic law and the associate plastic law. The equations are defined concretely using as the elastic law a simplified Signorini relation and the Prandtl–Reuss plastic relation.

The procedure of obtaining evolutionary constitutive equations for large elastoplastic strains described in [1] is a sort of formalization, an algorithm for deriving consistent (with the laws of thermodynamics and the objectivity principle) equations of state. The goal of the present study is to justify this procedure in deriving constitutive equations for finite elastic–inelastic (elastoplastic, viscoelastic, and thermoelastic) strains and to concretely define the obtained relations using the equations for a slightly compressible elastic material.

2. Kinematic Relations. Adhering to the approach described in [1] and based on the superposition of small strains on finite strains, we write the position gradient as the multiplication of small elastic, small inelastic, and finite elastic–inelastic strains (see Fig. 1):

$$F = f_E \cdot f_{IN} \cdot F_*. \tag{2.1}$$

Here the elastic–inelastic position gradient F_* transforms the initial configuration, in which the position of a point is specified by the radius-vector $\boldsymbol{r}$, to the first intermediate configuration $æ_1$. The gradient f_{IN} transforms the configuration $æ_1$ to the second, also intermediate, configuration $æ_2$, and the gradient f_E transforms the configuration $æ_2$ to the current configuration with the radius-vector $\boldsymbol{R}$. The configurations $æ_1$ and $æ_2$ and the current configuration are close to each other, which is formalized by the expressions

$$R_{æ_2} = R_{æ_1} + \varepsilon \boldsymbol{u}_{IN}, \qquad R = R_{æ_2} + \varepsilon \boldsymbol{u}_E, \tag{2.2}$$

where ε is a small (positive) parameter; $\boldsymbol{u}_{IN}$ and $\boldsymbol{u}_E$ are the inelastic and elastic displacement vectors that sequentially transform the configuration $æ_1$ to $æ_2$ and the configuration $æ_2$ to the current configuration. From relations (2.2), we obtain f_{IN} and f_E:

$$f_{IN} = (\overset{æ_1}{\nabla} R_{æ_2})^{\mathrm{t}} = g + \varepsilon h_{IN} = g + \varepsilon(e_{IN} + d_{IN}) = (g + \varepsilon e_{IN}) \cdot (g + \varepsilon d_{IN}),$$

$$f_E = (\overset{æ_2}{\nabla} R)^{\mathrm{t}} = g + \varepsilon h_E = g + \varepsilon(e_E + d_E) = (g + \varepsilon e_E) \cdot (g + \varepsilon d_E).$$

Here g is a unit tensor, e_{IN} and d_{IN} are the symmetric (small inelastic strains) and skew-symmetric (small inelastic rotations) parts of the tensor $h_{IN} = (\overset{æ_1}{\nabla} \boldsymbol{u}_{IN})^{\mathrm{t}}$, and e_E and d_E are the symmetric (small elastic strains) and skew-symmetric (small elastic rotations) parts of the tensor

$$\varepsilon h_E = \varepsilon(\overset{æ_2}{\nabla} \boldsymbol{u}_E)^{\mathrm{t}} = \varepsilon(f_{IN}^{-\mathrm{t}} \cdot \overset{æ_1}{\nabla} \boldsymbol{u}_E)^{\mathrm{t}} = \varepsilon(\overset{æ_1}{\nabla} \boldsymbol{u}_E)^{\mathrm{t}}.$$

From these expressions, it follows that the configurations $æ_1$ and $æ_2$ are indistinguishable (with accuracy up to a linear representation in ε):

$$f_E \cdot f_{IN} = f_{IN} \cdot f_E = g + \varepsilon(e + d) = (g + \varepsilon e) \cdot (g + \varepsilon d) = (g + \varepsilon d) \cdot (g + \varepsilon e),$$

where $e = e_E + e_{IN}$ is the total small strain and $d = d_E + d_{IN}$ is the total small rotation. As a result, relation (2.1) becomes

$$F = (g + \varepsilon h) \cdot F_* = [g + \varepsilon(e_E + e_{IN} + d_E + d_{IN})] \cdot F_*. \tag{2.3}$$

Here $h = h_E + h_{IN}$. The approximate relations (2.3) (which were obtained retaining only terms linear in ε) are easily made exact. Taking into account that $F - F_* = \Delta F$ is the position-gradient increment and $\varepsilon \boldsymbol{u} = \Delta \boldsymbol{u}$ is the displacement increment, we divide the first equation in relation (2.3) by Δt (the time of transition from the intermediate to the current configuration) and let the former to the latter (in this case, $\overset{æ_1}{\nabla}$ tends to $\tilde{\nabla}$ (the Hamiltonian in the current configuration). Finally, taking into account that $(\tilde{\nabla} \boldsymbol{v})^{\mathrm{t}} = \dot{F} \cdot F^{-1}$ is the displacement rate), we obtain an identity. Similarly, passing to the limit in the second equation of (2.3), we have

$$\dot{F} = (D_E + D_{IN} + W_E + W_{IN}) \cdot F = P \cdot F + Q \cdot F.$$

Here $D_E = \dot{e}_E$, $D_{IN} = \dot{e}_{IN}$ are the deformations of the elastic and inelastic displacement rates) (which in this case coincide with the corresponding strain rates); $W_E = \dot{d}_E$, $W_{IN} = \dot{d}_{IN}$ are the elastic and inelastic vorticity tensors; P and Q are arbitrary smooth tensor functions that satisfy the condition $P + Q = A$ ($A = D_E + D_{IN} + W_E + W_{IN}$).

The solution of the tensor equation (see [3])

$$\dot{F} = P(t) \cdot F(t) + Q(t) \cdot F(t) \tag{2.4}$$

for $F(t = 0) = g$ is the tensor (matriciant)

$$F_0^t(A) = F_0^t(P) \cdot F_0^t(S), \qquad S = [F_0^t(P)]^{-1} \cdot Q \cdot F_0^t(P), \tag{2.5}$$

where $F_0^t(P)$ and $F_0^t(S)$ are the solutions of Eq. (2.4) with right sides P and S (multiplied by F), respectively, with the same initial conditions, which is easy to verify by simple substitution. In solution (2.5), each of the matriciants is given by an expression of the form

$$F_0^t(A) = (g + A(\tau)\,\Delta t) \cdot F_0^{t_*}(A), \qquad \Delta t = t - t_*, \qquad \tau \in (t_*, t). \tag{2.6}$$

In a recursive extension, this is a product integral, which leads to a multiplicative integral in the limit $\Delta t \to 0$ [3].

Let us show that relations (2.5) and (2.3) are equivalent. Because $F(t)$ is $F_0^t(A)$, $P + Q = A$, taking into account the expression for the tensor S, using representation (2.6) for the matriciants, and retaining terms of only the first order of smallness in Δt, we reduce relation (2.5) to the form

$$(g + A(\tau)\,\Delta t) \cdot F_* = (g + A(\tau)\,\Delta t) \cdot F_0^{t_*}(P) \cdot F_0^{t_*}(S).$$

From this, $F_* = F_0^{t_*}(P) \cdot F_0^{t_*}(S)$, which should be the case according to relation (2.5), and $A(\tau)\,\Delta t = \varepsilon(e_E + e_{IN} + d_E + d_{IN})$.

Thus, choosing arbitrary smooth tensor functions P and Q with the specified sum, we obtain various multiplicative decompositions of the position gradient F. We set $P = D_E + W_E$. Then, according to (2.6), we have

$$F_0^t(P) = [g + \varepsilon(e_E + d_E)] \cdot F_0^{t_*}(P).$$

The last expression contains only elastic kinematics. Therefore, it is reasonable to introduce the notation $F_0^t(P) = F_E$ and the term the elastic position gradient:

$$F_E = [g + \varepsilon(e_E + d_E)] \cdot F_{E*}. \tag{2.7}$$

The tensor Q is determined by choosing the tensor P: $Q = D_{IN} + W_{IN}$. Then, using the second relation in (2.5), we obtain the tensor S, and, in accordance with (2.6), retaining terms only the first order of smallness in Δt or ε, we construct the tensor $F_0^t(S)$, which will be called the inelastic position gradient and denoted by F_{IN}:

$$F_{IN} = [g + \varepsilon F_{E*}^{-1} \cdot (e_{IN} + d_{IN}) \cdot F_{E*}] \cdot F_{IN*}. \tag{2.8}$$

As a result, from the first relation of (2.5), we obtain the representation $F = F_E \cdot F_{IN}$, which coincides in form with the well-known Lee decomposition but is free from the disadvantages of the latter. In particular, this representation implies that the total deformation of the displacement rate is the sum of the elastic and inelastic rate deformations and that the elastic position gradient does not vary for purely inelastic changes in the configuration.

The first statement follows from the following chain of relations. As is known, $l = (\tilde{\nabla} \boldsymbol{v})^{\mathrm{t}} = \dot{F} \cdot F^{-1}$. Then, $l = \dot{F}_E \cdot F_E^{-1} + F_E \cdot \dot{F}_{IN} \cdot F_{IN}^{-1} \cdot F_E^{-1}$. From this, using the relations $\dot{F}_E = (D_E + W_E) \cdot F_E$ and $\dot{F}_{IN} = F_E^{-1} \cdot (D_{IN} + W_{IN}) \cdot F_E \cdot F_{IN}$, which follow from expressions (2.7) and (2.8), we obtain $l = D_E + W_E + D_{IN} + W_{IN}$. Taking into account that the tensors D are symmetric and W are skew-symmetric, we have $D = (l + l^{\mathrm{t}})/2 = D_E + D_{IN}$.

To prove the second statement, we write the representation F as the multiplication of F_E and F_{IN} in terms of the corresponding Hamiltonian and radius-vectors in the well-known form

$$F = F_E \cdot F_{IN} = (\overset{IN}{\nabla} \boldsymbol{R})^{\mathrm{t}} \cdot (\nabla \boldsymbol{R}_{IN})^{\mathrm{t}} = \boldsymbol{R}_i \boldsymbol{R}_{IN}^i \cdot (\boldsymbol{R}_{IN})_j \boldsymbol{r}^j = \boldsymbol{R}_i \boldsymbol{r}^i. \tag{2.9}$$

Here the radius-vectors $\boldsymbol{R}$ and $\boldsymbol{R}_{IN}$ define the current and inelastic configurations, $\overset{IN}{\nabla}$ and ∇ are the Hamiltonians with respect to the inelastic and initial configurations, respectively, $\boldsymbol{R}_i$, $\boldsymbol{R}^i$, $(\boldsymbol{R}_{IN})_i$, $\boldsymbol{R}_{IN}^i$, $\boldsymbol{r}_i$, and $\boldsymbol{r}^i$ are the vectors of the principal and mutual bases of the current, inelastic, and initial configurations, respectively. According to (2.7), the elastic position gradient does not vary for purely inelastic changes in the configuration. From expressions (2.9) it follows that

$$F_E = (\overset{IN}{\nabla} \boldsymbol{R})^{\mathrm{t}} = \boldsymbol{R}_i \boldsymbol{R}_{IN}^i,$$

where and $\boldsymbol{R}$, and $\overset{IN}{\nabla}$, and hence, $\boldsymbol{R}_i$ and $\boldsymbol{R}_{IN}^i$, depend on plastic kinematics. Let us show that in this case there is no inconsistency.

The constancy of F_E for inelastic changes in the configuration is defined by the relation $\dot{F}_E = \dot{\boldsymbol{R}}_i \boldsymbol{R}_{IN}^i + \boldsymbol{R}_i \dot{\boldsymbol{R}}_{IN}^i = 0$. From this it follows that $\boldsymbol{R}^i \cdot \dot{\boldsymbol{R}}_i = \boldsymbol{R}_{IN}^i \cdot (\dot{\boldsymbol{R}}_{IN})_i$. Taking into account that the position of a point in the current configuration $\boldsymbol{R}$ is determined by its position in the previous close configuration $\boldsymbol{R}_*$ and by the vector of small inelastic displacements) $\varepsilon \boldsymbol{u}_{IN}$ ($\boldsymbol{R} = \boldsymbol{R}_* + \varepsilon \boldsymbol{u}_{IN}$), we have $\dot{\boldsymbol{R}}_i = \partial \boldsymbol{v}_{IN} / \partial q^i$, where $\boldsymbol{v}_{IN} = \dot{\boldsymbol{u}}_{IN}$ and q^i are generalized coordinates. As a result, we arrive at the following condition of constancy of the elastic position gradient for inelastic changes in the configuration: $\tilde{\nabla} \cdot \boldsymbol{v}_{IN} = \overset{IN}{\nabla} \cdot \dot{\boldsymbol{R}}_{IN}$.

Let us consider the factor in the square brackets in relation (2.8). This expression is the gradient of the vector $\boldsymbol{R}_{IN}$ relative to the intermediate inelastic configuration defined by the vector $\boldsymbol{R}_{IN*}$ and close to the current inelastic configuration:

$$\overset{IN*}{\nabla} \boldsymbol{R}_{IN} = \overset{IN*}{\nabla} (\boldsymbol{R}_{IN*} + \varepsilon \varphi) = g + \varepsilon \overset{IN*}{\nabla} \varphi,$$

where $\varepsilon \varphi$ is a small change in the plastic configuration. From this

$$\varepsilon \overset{IN*}{\nabla} \varphi = \varepsilon F_{E*}^{-1} \cdot h_{IN} \cdot F_{E*}.$$

Since $h_{IN} = (\overset{*}{\nabla} \boldsymbol{u}_{IN})^{\mathrm{t}}$ is the gradient of the vector of small inelastic displacements relative to the elastic–inelastic intermediate configuration [configuration æ$_1$ in relation (2.3)] which is close to the current configuration, we have $h_{IN} \cdot F_{E*} = (\overset{IN*}{\nabla} \boldsymbol{u}_{IN})^{\mathrm{t}}$. Then,

$$\varepsilon \overset{IN*}{\nabla} \varphi = \varepsilon F_{E*}^{-1} \cdot (\overset{IN*}{\nabla} \boldsymbol{u}_{IN})^{\mathrm{t}}.$$

Dividing the last relation by the increment in the time of the small inelastic process Δt and passing to the limit $\Delta t \to 0$, we obtain the equality $\overset{IN}{\nabla} \dot{\boldsymbol{R}}_{IN} = F_E^{-1} \cdot (\overset{IN}{\nabla} \boldsymbol{v}_{IN})^{\mathrm{t}}$, whose first invariant coincides with the above condition of independence of the elastic position gradient on plastic kinematics. Thus, relation (2.8) defines the variable inelastic configuration (the radius-vector $\boldsymbol{R}_{IN}$) relative to which the gradient of the vector $\boldsymbol{R}$ changed due to inelastic displacements remains constant. Of course, this elastic gradient can be written in any basis, including, for example, the variable basis included in relation (2.9).

Thus, using expressions (2.1) or (2.3), which and only which define the true history of deformation, we obtained a multiplicative decomposition of the total position gradient into an elastic gradient and an inelastic gradient, which coincides in shape with the well-known Lee decomposition but has a completely different nature. Naturally, the thus obtained elastic and inelastic position gradients [relations (2.7) and (2.8)] mirror the true history of deformation: the substitution of the expressions of these gradients into the right side of the Lee decomposition yields relations (2.1) or (2.3).

In some cases, it is useful to treat the representation F in the form of the multiplication F_E and F_{IN} as a series connection of an elastic and an inelastic element, using the nomenclature of structural modeling. Indeed, within the framework of small strains ($F_{E*} = F_{IN*} = g$), relations (2.7) and (2.8) imply that the total strain and the total rotation are equal to the sum of elastic and inelastic quantities, which corresponds to a series connection of an elastic and an inelastic elements.

The correct isolation of the purely elastic term from the elastic–inelastic position gradient performed here will be needed to us to construct constitutive equations for the elastic–inelastic behavior of the material at finite strains. We shall also need the Cauchy–Green strain measure $C = F^{\mathrm{t}} \cdot F$, which, taking into account expressions (2.3) and (2.5) is written as

$$C = F^{\mathrm{t}}_{IN} \cdot C_E \cdot F_{IN} = C_* + 2\,\varepsilon F^{\mathrm{t}}_* \cdot (e_E + e_{IN}) \cdot F_*, \tag{2.10}$$

where $F_* = F_{E*} \cdot F_{IN*}$.

3. Constitutive Equation. Any elastic–inelastic process that results in the current configuration æ is treated as an elastic process from the strained configuration $æ_2$ close to the current configuration (see Fig. 1). The closeness is due to the possibility of using relation (1.3) as the constitutive elastic equation, which admits a convenient treatment related to the terms containing the stress T_* corresponding to the position gradient F_*. These terms completely determine the transformation of the stress T_* upon superimposition of the position gradient f on F_*, i.e., the rotation of this stress and its changes due to a change in the current elementary area. Indeed, the oriented elementary areas in the current and intermediate configurations are linked by the well-known relation (see [1]) $J_f^{-1} N \cdot f\, dS = N_*\, dS_*$, where $J_f = I_3(f) = 1 + \varepsilon I_1(e)$ and $N\, dS$ and $N_*\, dS_*$ are the oriented elementary areas in the current and intermediate ($æ_2$) configurations. Setting the strain corresponding to the stress T_* in the intermediate configuration equal to the strain in the current configuration, we obtain the symmetric stress tensor $J_f^{-1}\, f \cdot T_* \cdot f^{\mathrm{t}}$ for the latter. Substituting the expressions for J_f and f into the above relation and retaining only terms linear in ε, we arrive at the relation $[1 - I_1(e)]T_* + h \cdot T_* + T_* \cdot h^{\mathrm{t}}$, in which, taking into account that $h = e + d$, it is possible to distinguish terms that are due only to the rotation and variation in the value of the area.

Let us consider the transition from the configuration $æ_1$ to the current configuration æ (see Fig. 1). The configuration $æ_1$ corresponds to the accumulated stress state $T_{æ_1}$. The gradient f_{IN} transforms the configuration $æ_1$ with this stress state to the configuration $æ_2$ by rotating the tensor $T_{æ_1}$ by means of d_{IN} and converting it to the new current area by means of e_{IN} (path 1 in Fig. 1). As a result, the stress state in the configuration $æ_2$ is given by the relation

$$T_{æ_2} = [1 - \varepsilon I_1(e_{IN})]T_{æ_1} + \varepsilon h_{IN} \cdot T_{æ_1} + \varepsilon T_{æ_1} \cdot h^{\mathrm{t}}_{IN}. \tag{3.1}$$

The stress $T_{æ_2}$ is the initial one for the kinematics defined by the elastic position gradient f_E; therefore, according to the constitutive equation (1.3), the true stress tensor is written as

$$T = [1 - \varepsilon I_1(e_E)]\, T_{æ_2} + \varepsilon h_E \cdot T_{æ_2} + \varepsilon T_{æ_2} \cdot h^{\mathrm{t}}_E + \varepsilon \tilde{L}^{IV}_6 \cdot\cdot\, e_E, \tag{3.2}$$

where $\tilde{L}^{IV}_6$ defines the material response to small elastic strains e_E relative to the configuration $æ_2$. Substituting expression (3.1) into relation (3.2) and retaining only terms linear in ε, we arrive at the equation

$$T = [1 - \varepsilon I_1(e)]\, T_{æ_1} + \varepsilon h \cdot T_{æ_1} + \varepsilon T_{æ_1} \cdot h^{\mathrm{t}} + \varepsilon \tilde{L}^{IV}_6 \cdot\cdot\, e_E, \tag{3.3}$$

where $e = e_E + e_{IN}$ and $h = h_E + h_{IN}$ are the total small strain and the total displacement gradient for the transition from the configuration $æ_1$ [with stress $T_{æ_1}$ (below T_*) accumulated in it] to the current configuration. [We note that relation (3.3) can be obtained following path 2 (see Fig. 1), i.e., by first performing a small elastic process and then a small inelastic process, as follows from the above equality $f_E \cdot f_{IN} = f_{IN} \cdot f_E$.] Because $e_E = e - e_{IN}$, it follows that

$$T = [1 - \varepsilon I_1(e)]T_* + \varepsilon h \cdot T_* + \varepsilon T_* \cdot h^{\mathrm{t}} + \varepsilon \tilde{L}_6^{IV} \cdot\cdot (e - e_{IN}). \tag{3.4}$$

The approximate equation (3.4) can be reduced an exact (differential and evolutionary) equation by dividing it by the increment in the time of transition from the configuration $æ_1$ to the actual configuration and letting the intermediate configurations to the current configuration ($æ_1 \to æ_2 \to æ$). As a result, we have

$$T^{\mathrm{Tr}} = \tilde{L}_6^{IV} \cdot\cdot (\dot{e} - \dot{e}_{IN}), \tag{3.5}$$

where $T^{\mathrm{Tr}} \equiv \dot{T} - \dot{h} \cdot T - T \cdot \dot{h}^{\mathrm{t}} + I_1(\dot{e})T$ is the resulting Truesdell objective derivative, $\dot{h} \equiv (\tilde{\nabla}\boldsymbol{v})^{\mathrm{t}}$, and $\dot{e} \equiv D$ (D is the tensor of the total-displacement rate deformation). The arguments of the fourth-rank tensor $\tilde{L}_6^{IV}$ in (3.5) are the same kinematic quantities as in relation (3.4) but they are in the current rather than intermediate configuration. As a result, introducing the equation of state for $\dot{e}_{IN}$, we obtain an evolutionary constitutive equation written in terms of the true stress, the stress rate, the total-displacement rate deformation, and the elastic kinematics defining the tensor $\tilde{L}_6^{IV}$. Let us construct a particular expression for this tensor.

4. Elastic Potential. We consider a purely elastic material whose behavior is determined by purely elastic kinematics. (Below, for simplicity, we omit the subscript E at all kinematic quantities, assuming that they are elastic.)

The true stress tensor is written in terms of the Piola–Kirchhoff tensor of the second order as $T = J^{-1} F \cdot P_{II} \cdot F^{\mathrm{t}}$. A comparison of this expression with relation (1.2) shows that $\tilde{g}_6(U) = P_{II}$. The tensor P_{II} is defined in elastic theory using the elastic potential W, which depends, as a rule, on the Cauchy–Green elastic-strain measure $C = F^{\mathrm{t}} \cdot F$: $P_{II} = 2\,(\partial W/\partial C)$. Using the well-known rules of differentiation of tensor functions of a tensor argument with respect to the tensor argument [4], we have

$$dP_{II} = \frac{\partial P_{II}}{\partial C} \cdot\cdot dC = \frac{\partial P_{II}}{\partial C} \cdot\cdot (2F^{\mathrm{t}} \cdot de \cdot F) = 2\Big(F \overset{3}{\circ} \frac{\partial P_{II}}{\partial C} \cdot F^{\mathrm{t}}\Big) \cdot\cdot de.$$

Here we took into account that according to relation (2.10) (in which $F_{IN} = g$ and $e_{IN} = 0$), in the limit (letting the intermediate configuration to the current configuration), we have $dC = 2F^{\mathrm{t}} \cdot de \cdot F$. The expression $A \overset{3}{\circ} B^{IV}$ denotes positional multiplication, i.e., the scalar premultiplication of the second-rank tensor A by the third basis vector of the fourth-rank tensor B^{IV}. Now, using the representation of P_{II} in terms of the elastic potential, we obtain

$$P_{II} = 4 \int\limits_{e_0^t} \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot de = 4 \int\limits_0^t \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot D\,d\tau. \tag{4.1}$$

Here e_0^t is the history of the tensor e, and all integrands in the last integral are functions of time τ

Similarly,

$$dW = \frac{\partial W}{\partial C} \cdot\cdot dC = \frac{\partial W}{\partial C} \cdot\cdot (2F^{\mathrm{t}} \cdot de \cdot F) = 2\Big(F \cdot \frac{\partial W}{\partial C} \cdot F^{\mathrm{t}}\Big) \cdot\cdot de = (F \cdot P_{II} \cdot F^{\mathrm{t}}) \cdot\cdot de.$$

From this,

$$W = \int\limits_{e_0^t} (F \cdot P_{II} \cdot F^{\mathrm{t}} \cdot\cdot de = \frac{1}{2} \int\limits_{C_0^t} P_{II} \cdot\cdot dC, \tag{4.2}$$

where C_0^t is the history of the tensor C, or, taking into account (4.1), we have

$$W = 4 \int\limits_{e_0^t} \Big\{F \cdot \Big[\int\limits_{e_0^t} \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot de\Big] \cdot F^{\mathrm{t}}\Big\} \cdot\cdot de = 2 \int\limits_{C_0^t} \Big[\int\limits_{e_0^t} \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot de\Big] \cdot\cdot dC$$

$$= 4 \int\limits_0^t \Big\{F \cdot \Big[\int\limits_0^{\tau_1} \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot D\,d\tau_2\Big] \cdot F^{\mathrm{t}}\Big\} \cdot\cdot D\,d\tau_1. \tag{4.3}$$

The last equality in (4.3) implies that

$$\frac{dW}{dt} = P_{II} \cdot\cdot (F^{\mathrm{t}} \cdot D \cdot F) = JT \cdot\cdot D.$$

Taking into account that $D\,dt = de$ and $2F^{\mathrm{t}} \cdot de \cdot F = dC$, we obtain the relations $P_{II} = 2\,(\partial W/\partial C)$ and $T = J^{-1}\,(\partial W/\partial e)$. The last expressions also follow from the other equalities in (4.2) and (4.3). Determining the second derivative with respect to C from the second equality in (4.3), we obtain an identity. This form of the elastic potential allows it to be extended to an elastic–inelastic process.

As follows from relation (4.1), the fourth-rank tensor $\partial^2 W/\partial C^2$ defines the properties of the material at the current time relative to the basis of the initial configuration and its response (stress referred to the initial configuration) to an infinitesimal increment of the Cauchy–Green strain measure. The tensor F simply transforms the last two basis vectors of this fourth-rank tensor to the current-configuration vectors, to which the tensor de is referred. Let us model an elastic–inelastic process using a series connection of an elastic and an inelastic elements whose kinematics is given by relations (2.7) and (2.8), respectively; the kinematics of the entire assembly is defined by (2.5). For this connection of elements, the true stresses in them are identical and equal to the total stress of the entire assembly. The increment of these stresses is determined by the properties of the elastic element, i.e., by the fourth-rank tensor $\partial^2 W(C_E)/\partial C_E^2$, and the elastic-strain increment. Therefore, a natural extension of relations (4.1) and (4.3) to the elastic–inelastic process are the relations

$$P_{II} = 4\int_0^t \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot\, D_E\, d\tau; \tag{4.4}$$

$$W_1 = 4\int_0^t \Big\{F \cdot \Big[\int_0^{\tau_1} \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot\, D_E\, d\tau_2\Big] \cdot F^{\mathrm{t}}\Big\} \cdot\cdot\, D_E\, d\tau_1, \tag{4.5}$$

where F is the total elastic–inelastic position gradient and W is the elastic potential that depends only on the elastic kinematics defined by expression (2.7). These relations imply that

$$JT \cdot\cdot\, D_E = \frac{dW_1}{dt} = 4\Big\{F \cdot \Big[\int_0^t \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot\, D_E\, d\tau\Big] \cdot F^{\mathrm{t}}\Big\} \cdot\cdot\, D_E. \tag{4.6}$$

From this, we have

$$T = 4J^{-1} F \cdot \Big[\int_0^t \Big(F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^{\mathrm{t}}\Big) \cdot\cdot\, D_E\, d\tau\Big] \cdot F^{\mathrm{t}}.$$

The last integral can be written as the sum of two integrals: the first from 0 to t_*, which, according to (4.4), is P_{II*}, and the second from t_* to t, where $t - t_* = \Delta t$ is a small finite quantity; therefore, it can be approximated by the expression

$$\Big(F_* \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2}\Big|_{C_E = C_{E*}} \cdot F_*^{\mathrm{t}}\Big) \cdot\cdot\, \varepsilon e_E.$$

Here and below, the quantities with the subscript "$*$" refer to the time t_*. Now, taking into account relation (2.3) for F and the representation $J = J_*(1 + \varepsilon I_1(e))$ for the Jacobian [whence $J^{-1} = J_*^{-1}(1 - \varepsilon I_1(e))$] [1] and retaining only terms linear in ε, we arrive at the constitutive equation (3.3) with the particular expression for: $\tilde L_6^{IV}$:

$$\tilde L_6^{IV} = 4J_*^{-1} F_* \cdot \Big(F_* \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2}\Big|_{C_E = C_{E*}} \overset{2}{*} F_*^{\mathrm{t}}\Big) \cdot F_*^{\mathrm{t}}, \tag{4.7}$$

where $B^{IV} \overset{2}{*} A$ denotes the scalar multiplication of the second-rank tensor A by the second basis vector of the fourth-rank tensor B^{IV}.

5. Elastic Potential *W* of a Slightly Compressible Material. Since the potential W in (4.4)–(4.7) depends on the only elastic strain measure, we omit the subscript E in the notation of kinematic quantities at the beginning of this section.

Elastomers, i.e., materials that can be deformed to large elastic strains, exhibit slight compressibility and are considered incompressible under normal conditions. This approximation is adopted by most researchers and is quite correct for materials operating under ordinary (not extreme) conditions, i.e., at a low hydrostatic pressure [5–7]. In modern engineering facilities, elastomers operate under extreme conditions at high hydrostatic pressures due, in particular, to temperature fields, and neglect of the slight compressibility of the material leads to unrealistic results. The theory of finite elastic strains of an initially isotropic, slightly compressible material developed in [8] is based on a series expansion of the elastic potential W in the third principal invariant of the Cauchy–Green strain measure C in the neighborhood of unity with retention of only terms not higher than quadratic terms (by virtue of the slight compressibility of the material):

$$W(I_1, I_2, I_3) = \hat{W}(I_1, I_2) + \chi_1(I_3 - 1) + (1/2)\chi_2(I_3 - 1)^2,$$

$$\hat{W}(I_1, I_2) = W(I_1, I_2, 1), \qquad \chi_1(I_1, I_2) = \frac{\partial W}{\partial I_3}\Big|_{I_3=1}, \qquad \chi_2(I_1, I_2) = \frac{\partial^2 W}{\partial I_3^2}\Big|_{I_3=1}.$$

Here $I_i = I_i(C)$ $(i = 1, 2, 3)$ are the principal invariants of the measure C. This introduces four generalized elastic moduli, one of which defines the compressibility (incompressibility) of the material.

The theory is further developed in studies of [9–11], which take into account the experimentally observed variation in the bulk modulus and shear modulus due to volume variation and demonstrate the associated effects, in particular, the untwisting and stretching of the previously wound and axially compressed outer surface of a hollow cylinder by supplying internal pressure. For finite strains, the expressions defining the shear modulus and the bulk modulus depend on the particular problem, i.e., they are different for different problems (as is described in [8, 9]). This is not surprising since the hydrostatics for finite strains is determined by both volume and shape variations, and, vice versa, the volume variation is determined by both the spherical and deviator parts of the stress tensor. In this connection, the terms the bulk modulus and the shear modulus are purely conditional and terminologically convenient, but they should necessarily be assigned to a particular problem; therefore, they are placed in quotations.

The constitutive relation obtained and used in [9–11] has the form

$$\frac{1}{2}P_{II} = \frac{\partial W}{\partial C} = (g - I_3C^{-1})c_1 + (I_1g - C - 2I_3C^{-1})c_2 + \sigma I_3C^{-1},$$

$$\alpha(\sigma - \chi_1) = I_3 - 1, \qquad \alpha = 1/\chi_2, \tag{5.1}$$

$$c_i(\hat{I}_1, \hat{I}_2, I_3) = k_i + p_i(I_3 - 1) + (1/2)q_i(I_3 - 1)^2,$$

$$k_i(\hat{I}_1, \hat{I}_2) = \frac{\partial \hat{W}}{\partial \hat{I}_i}, \qquad p_i(\hat{I}_1, \hat{I}_2) = \frac{\partial \chi_1}{\partial \hat{I}_i}, \qquad q_i(\hat{I}_1, \hat{I}_2) = \frac{\partial \chi_2}{\partial \hat{I}_i} \qquad (i = 1, 2).$$

Here

$$\hat{I}_1 = I_1 - (I_3 - 1), \qquad \hat{I}_2 = I_2 - 2(I_3 - 1), \qquad \hat{I}_3 = I_3 \tag{5.2}$$

are the invariants introduced in [12]. Furthermore,

$$\chi_1(\hat{I}_1, \hat{I}_2)\Big|_{C=g} = 0, \qquad 2(k_1 + k_2)\Big|_{C=g} = G_0,$$

$$4\Big(\chi_2 - \frac{2(k_1 + k_2)}{3}\Big)\Big|_{C=g} = B_0, \qquad \frac{\chi_2 - (k_1 + k_2)}{2\chi_2 - (k_1 + k_2)}\Big|_{C=g} = \nu_0,$$

where G_0, B_0, and ν_0 are the shear modulus, the bulk modulus, and Poisson's ratio of the linear theory of elasticity. In order that the constitutive equations be energetically permissible [13], the following equalities should be satisfied:

$$\frac{\partial k_1}{\partial \hat{I}_2} = \frac{\partial k_2}{\partial \hat{I}_1}, \qquad \frac{\partial p_1}{\partial \hat{I}_2} = \frac{\partial p_2}{\partial \hat{I}_1}, \qquad \frac{\partial q_1}{\partial \hat{I}_2} = \frac{\partial q_2}{\partial \hat{I}_1}.$$

In [9–11], $\hat{W}$, χ_1, and χ_2 were specified by the elementary expressions

$$\hat{W} = k_1(\hat{I}_1 - 3) + k_2(\hat{I}_2 - 3), \qquad \chi_1 = p_1(\hat{I}_1 - 3) + p_2(\hat{I}_2 - 3),$$

$$\chi_2 = \chi_{20} + q_1(\hat{I}_1 - 3) + q_2(\hat{I}_2 - 3) = \chi_{20}(1 + Q_1(\hat{I}_1 - 3) + Q_2(\hat{I}_2 - 3)), \qquad Q_i = q_i/\chi_{20}$$

$$(k_1,\ k_2,\ p_1,\ p_2,\ q_1,\ q_2,\ \chi_{20} \text{ are constants}),$$

which nevertheless revealed the effect related to the slight elastic compressibility of the material.

From relations (5.1), it is easy to obtain the following expression for $\partial^2 W/\partial C^2$ taking into account that (see [4, 13])

$$\frac{\partial I_1}{\partial C} = g, \qquad \frac{\partial I_2}{\partial C} = I_1 g - C, \qquad \frac{\partial I_3}{\partial C} = I_3 C^{-1},$$

$$\frac{\partial C}{\partial C} = C_{II}^{IV}, \qquad \frac{\partial C^{-1}}{\partial C} = -C^{-1} \cdot C_{II}^{IV} \overset{2}{*} C^{-1}.$$

Here C_{II}^{IV} is the second isotropic tensor of the fourth rank. As a result, taking into account (4.7) and again denoting the purely elastic quantities by the subscript E, we obtain

$$\tilde{L}_6^{IV} \cdot\cdot\, e_E = 4J_*^{-1}\{(c_1 - \sigma_*)I_{3E*}[Y \cdot e_E \cdot Y - Y(Y \cdot\cdot\, e_E)]$$

$$+ I_{3E*}[Y(\Phi_* \cdot\cdot\, e_E) + (\Phi_* - 2I_{3E*}Y)(Y \cdot\cdot\, e_E)][p_1 + q_1(I_{3E*} - 1)]$$

$$+ [\Phi_*(\Phi_* \cdot\cdot\, e_E) - \Phi_* \cdot e_E \cdot \Phi_* + 2I_{3E*}(Y \cdot e_E \cdot Y - (Y \cdot\cdot\, e_E)Y]c_2$$

$$+ I_{3E*}[I_{1E*}Y(\Phi_* \cdot\cdot\, e_E) - Y(X \cdot\cdot\, e_E) + (I_{1E*}\Phi_* - X - 4I_{3E*}Y)(Y \cdot\cdot\, e_E)][p_2 + q_2(I_{3E*} - 1)]$$

$$+ I_{3E*}^2[\chi_{20} + q_1(\hat{I}_{1E*} - 3) + q_2(\hat{I}_{2E*} - 3)]Y(Y \cdot\cdot\, e_E)\}. \tag{5.3}$$

Here $c_i = k_i + p_i(I_{3E*} - 1) + (1/2)q_i(I_{3E*} - 1)^2$ $(i = 1, 2)$, σ_* is given by the relation

$$\alpha_0 \frac{\sigma_* - p_1(\hat{I}_{1E*} - 3) - p_2(\hat{I}_{2E*} - 3)}{1 + Q_1(\hat{I}_{1E*} - 3) + Q_2(\hat{I}_{2E*} - 3)} = I_{3E*} - 1, \qquad \alpha_0 = \frac{1}{\chi_{20}}, \tag{5.4}$$

$\hat{I}_{1E}$, $\hat{I}_{2E}$ are defined by relations (5.2), $Y = F_* \cdot C_{E*}^{-1} \cdot F_*^{\mathrm{t}}$, $X = F_* \cdot C_{E*} \cdot F_*^{\mathrm{t}}$, and $\Phi_* = F_* \cdot F_*^{\mathrm{t}}$ is the Finger strain measure tensor, whose invariants coincide with the corresponding invariants of the tensor C_*. All quantities with the subscript "$*$" are referred to the attained intermediate configuration æ$_1$ and are therefore known. In relations (5.3), representing the small elastic strains as the difference of small total and inelastic strains (for the latter, one needs to write their particular equations of state, some of which are given in Sec. 4), we complete the derivation of the constitutive equation (3.4), which describes elastic–inelastic material behavior at finite strains, finite elastic (slightly compressible materials), and inelastic strains. The total kinematics present in relations (3.4) and (5.3) is defined by expression (2.3), and the elastic kinematics in (5.3) and (5.4) by expression (2.7).

Conclusions. In the kinematics of an elastic–inelastic process, purely elastic kinematics independent of inelastic changes in the strain configuration was distinguished. An evolutionary equation (with the resulting objective derivative) for the behavior of complex media at finite strains was derived treating an elastic–inelastic process as an elastic process from a uniquely determined stressed configuration close to the current configuration. The procedure of deriving the equation is easily formalized and was used to construct the constitutive relations of elastoplasticity [1], viscoelasticity, and thermoelasticity [14, 15] at finite strains.

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